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Quantum Logics Derived from Asymmetric Mielnik Forms

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It is shown that a logic will possess a "rich" set of states if and only if it can be derived from a Mielnik form, not necessarily symmetric.

1. INTRODUCTION

The question of recapturing the set of all events in a quantum logic $\mathcal{L}$ from the set $\mathfrak{M}$ of all states, or from a suitable subset, is of some importance, because on the one hand it offers the possibility of systematic construction of logics, while on the other reduces the study of $\mathcal{L}$ to that of $\mathfrak{M}$, which in general appears to have some advantages.

There is a wide class of logics for which this is possible. They are characterized by the list of properties to be described below under the name of axioms, all of which seem to possess a reasonable physical interpretation.

By a logic we mean a set $\mathcal{L}$ partially ordered by a relation $\leqslant$ and carrying a map $'$: $\mathcal{L} \to \mathcal{L}$, for which we assume the following:

Axiom 1. For all A, B we have $A \leqslant B$ implies $B' \leqslant A'$, while $(A')' = A$.

Axiom 2. With $\wedge, \vee$ denoting infinum and supremum relative to the order $\leqslant$ (not assumed to exist universally) we have $A \wedge A' = 0$, $A \vee A' = I$ for two fixed elements O and I of $\mathcal{L}$ (and all $A \in \mathcal{L}$), while $O' = I$, $I' = 0$.

Axiom 3. The orthomodular law holds: If $A \leqslant B$ then $A' \wedge B$ exists and $B = A \vee (A' \wedge B)$.

Elements A, B for which $A \leqslant B'$ (or equivalently $B \leqslant A'$) are called disjoint. We write this as $A \perp B$.

Axiom 4. Infinite disjoint suprema exist: If $A_i \leqslant A_j'$ for $i \neq j$ then the supremum ΣA_i of the family $\langle A_i \rangle$ exists.

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We should note that this assumption is stronger than the usual one, which postulates the existence of suprema only for countable pairwise disjoint families. We should, however, also point out that the standard Hilbert space models do satisfy our stronger hypothesis, and that in general, if the logic is separable, Axiom 4 follows from the usual countable hypothesis.

Needless to say, any attempt toward answering the problem posed at the beginning of this section is going to fail, unless "enough" states of $\mathcal{L}$ exist.

A *state* is a map m: $\mathcal{L} \to [0,1]$ such that $m(I)=1$, and for any family $\{A_i\}$ of pairwise disjoint elements we have $m(\Sigma A_i)=\Sigma m A_i$.

Again, our definition is stronger than usual, since it involves arbitrary, rather than countable, suprema. But, as noted above, for the classical models any state in the usual sense is also a state in our sense; for the general separable logic there is also no distinction.

We shall assume the existence of a set $\mathfrak{M}$ of states which is "rich" as expressed in the following assumptions.

Axiom 5. If $pB=1$ for each $p \in \mathfrak{M}$ for which $pA=1$, then $A \leqslant B$.

Axiom 6. For each $p \in \mathfrak{M}$ the element $L_p = \inf\{A \mid pA=1\}$ exists and $p(L_p)=1$. Conversely, if $A \neq 0$, there exists a $p \in \mathfrak{M}$ with $pA=1$ (i.e., $L_p \leqslant A$).

We call L_p the *support* of the state p.

Given any two elements $p, q \in \mathfrak{M}$, we can define the probability of transition from p to q as the number $p(L_q)$, which we shall write as $(p \to q)$. Note that since L_q is the cause of all events that occur with certainly in q, this definition does make sense. We shall *not* assume that $(p \to q)$ is symmetric in its variables (although we shall not exclude this case). Thus our system of transition probabilities will not form a classical Mielnik system.

In the next section we shall establish the basic properties of this functional $(p \to q)$ which will form the foundation of the construction presented later. For the present we confine ourselves to the remark that the set $\mathfrak{M}$ equipped with this functional is quite sufficient to reproduce $\mathcal{L}$ completely.

2. PRELIMINARY RESULTS

We begin with the following remark.

Lemma 1. For any $p, q \in \mathfrak{M}$ we have $L_p \perp L_q$ iff $p(L_q)=0$ iff $q(L_p)=0$.

Proof. Since $L_p \perp L_q$ iff $L_p \leqslant L'_q$, we have $L_p \perp L_q$ iff $p(L'_q) = 1$, i.e., iff $p(L_q) = 0$. Since $L_p \perp L_q$ is symmetric in p, q the rest follows. ■

Proposition 1. If $A \neq 0$ then A is the (disjoint) sum of supports.

Proof. Since by Axiom 6 there exists a $p \in \mathfrak{M}$ for which $L_p \leqslant A$, we can obtain by Zorn's lemma a maximal family $\{L_{p_i}\}$ with $L_{p_i} \perp L_{p_j}$ (for $i \neq j$) and $L_{p_i} \leqslant A$. By Axiom 4 the event $B = \Sigma L_{p_i}$ exists. If $B \neq A$, then $A \wedge B' \neq 0$ (by orthomodularity), hence there is a $q \in \mathfrak{M}$ with $q(A \wedge B') = 1$ and so $L_q \leqslant A$. But then, since $L_{p_i} \perp A \wedge B'$, we have $q(L_{p_i}) = 0$, or $L_q \perp L_{p_i}$; thus $\{L_{p_i}\}$ is not maximal disjoint—a contradiction. Thus $B = A$, or $A = \Sigma L_{p_i}$ with L_{p_i} pairwise disjoint. ■

Proposition 2. An event A is completely determined by $\{p \in \mathfrak{M} \mid pA = 1\}$.

Proof. This is immediate by Axiom 5. ■

The question of characterizing "intrinsically" all sets of the form $\{p \in \mathfrak{M} \mid pA = 1\}$ is, of course, our original question. We shall formulate the answer in terms of the functional $(p \to q)$, whose properties we first have to obtain.

Proposition 3. The transition probability functional has the following properties:

(i) $(p \to p) = 1$ for any $p \in \mathfrak{M}$.
(ii) $(p \to q) = 0$ implies $(q \to p) = 0$ for all $p, q \in \mathfrak{M}$.
(iii) If $\{p_i\}$ is a family in $\mathfrak{M}$ such that, for $i \neq j$, $(p_i \to p_j) = 0$ while for any p not in the family we have at least one i with $(p \to p_i) \neq 0$—in short, if $\{p_i\}$ is a maximal "orthogonal" family in $\mathfrak{M}$—then for all $p \in \mathfrak{M}$ we have $\Sigma(p \to p_i) = 1$.

Proof. Part (i) follows from Axiom 6, since $p(L_p) = 1$. Part (ii) is just Proposition 1. For part (iii), note that for such a family $\{p_i\}$ we have $\Sigma L_{p_i} = I$. But then $1 = p(I) = \Sigma p(L_{p_i}) = \Sigma(p \to p_i)$.

Note. For any $A \in \mathfrak{L}$, if we consider a maximal disjoint family $\{L_{p_i}\}$ of supports such that $A = \Sigma L_{p_i}$, then $pA = \Sigma(p \to p_i)$, according to Proposition 1.

3. THE CHARACTERIZATION OF $\mathfrak{L}$

We shall use the term *span* to describe a set S such that, for some orthogonal set $\{p_i\} \subseteq S$ we have $p \in S$ iff $\Sigma(p \to p_i) = 1$.

Theorem 1. For any $A \in \mathfrak{L}$ the set $\{p \in \mathfrak{M} \mid pA = 1\}$ is a span and conversely, every span has this form for some unique event $A \in \mathfrak{L}$.

Proof. Given A consider any maximal disjoint family of supports, say, $\{L_{p_i}\}$, with $A=\Sigma L_{p_i}$. Then, by the note at the end of the previous section, we have $pA=1$ iff $\Sigma(p\to p_i)=1$. Thus the first part is established. Conversely let S be a span, and let $\{p_i\}$ be an orthogonal family such that $p\in S$ iff $\Sigma(p\to p_i)=1$. The events L_{p_i} being pairwise disjoint, we set $A=\Sigma L_{p_i}$ and note that $pA=1$ iff $\Sigma p(L_{p_i})=1$, i.e., iff $p\in S$. Clearly this A is unique by Proposition 2. ■

We can describe the partial order $\leqslant$ and the complementation in terms of spans as follows.

> *Theorem 2.* If A corresponds to the span S and B to the span T, then $A\leqslant B$ iff $S\subseteq T$. Further the span corresponding to A' is $\{p|(p\to q)=0 \text{ for all } q\in S\}$.

Proof. The first part is just Axiom 5. The second follows from the obvious fact that $p(A')=1$ iff $p(A)=0$: because $q\in S$ iff $qA=1$ iff $L_q\leqslant A$; hence $p(A')=1$ implies $p(L_q)=0$, i.e., $(p\to q)=0$. Thus every p in the span corresponding to A' is orthogonal to each state in S. Conversely, if $(p\to q)=0$ for all $q\in S$, then $pA=0$, because by Proposition 1 the event A has the form ΣL_{p_i} with $p_i\in S$, hence $pA=\Sigma(p\to p_i)=0$. ■

These two theorems provide the answer to our original question of characterizing the events in terms of the state space $\mathfrak{M}$. We shall now proceed with the construction promised in the introduction.

4. GENERALIZED MIELNIK SYSTEMS

We consider a set $\mathfrak{M}$ and a map from $\mathfrak{M}\times\mathfrak{M}$ to $[0,1]$, whose value at (p,q) we shall write as $(p\to q)$ to emphasize its interpretation as the probability of transition from p to q. We do not assume symmetry; all we need is contained in Proposition 3, but we repeat it here for the reader's convenience.

(i) For all $p\in\mathfrak{M}$ we have $(p\to p)=1$.
(ii) If $(p\to q)=0$, then $(q\to p)=0$ also. Thus we can refer to such p,q as being orthogonal, without ambiguity.
(iii) If $\{p_i\}$ is a maximal orthogonal family in $\mathfrak{M}$, then for any $p\in\mathfrak{M}$ we have $\Sigma(p\to p_i)=1$.

The usual two extra conditions associated with Mielnik systems, namely, $(p\to q)=1$, implies $p=q$ and $(p\to q)=(q\to p)$, will not be assumed, as they are not needed. Neither of these follows from (i), (ii), (iii) above, as the following example shows:

$\to$	p	q	r	s
p	1	0	$\frac{1}{2}$	$\frac{1}{2}$
q	0	1	0	0
r	1	0	1	0
s	1	0	0	1

We shall refer to a family of the form $\{(p \to q) | q \in \mathfrak{M}\}$ as a "row" and to one of the form $\{(p \to q) | p \in \mathfrak{M}\}$ as a "column." It is quite possible for two columns to be identical while the corresponding rows are distinct:

$\to$	p	q	r	s	t
p	1	1	0	$\frac{1}{2}$	$\frac{1}{2}$
q	1	1	0	$\frac{1}{3}$	$\frac{2}{3}$
r	0	0	1	0	0
s	1	1	0	1	0
t	1	1	0	0	1

As we shall see later, however, if two rows are identical, the two corresponding columns are also identical, in which case we shall delete one of these two rows and the corresponding column, as it merely contributes unnecessary duplication.

We shall retain the use of the term *span* introduced earlier: if $\{p_i\}$ is any orthogonal family in $\mathfrak{M}$ we shall write $\mathrm{SP}\{p_i\}$ for the set $\{p | \Sigma(p \to p_i) = 1\}$. For any $S \subseteq \mathfrak{M}$ we shall write $S^\perp$ for $\{q | (p \to q) = 0$ for all $p \in S\}$.

Lemma 2. Let $\{p_i\}$ be pairwise orthogonal; then $\mathrm{SP}\{p_i\} = (\{p_i\}^\perp)^\perp$.

Proof. Consider any r orthogonal to each element $q \in \{p_i\}^\perp$, and augment (by Zorn's lemma) the family $\{p_i\}$ by elements $\{q_j\}$ to obtain a maximal orthogonal family $\{p_i, q_j\}$ in $\mathfrak{M}$. Then $\Sigma(r \to p_i) + \Sigma(r \to q_j) = 1$; but $q_j \in \{p_i\}^\perp$ hence $(r \to q_j) = 0$ and so $\Sigma(r \to p_i) = 1$, i.e., $r \in \mathrm{SP}\{p_i\}$. For the reverse, let $q \in \{p_i\}^\perp$ and augment by the elements $\{r_k\}$ to obtain a maximal orthogonal family $\{p_i, q, r_k\}$ in $\mathfrak{M}$. For any r we have $\Sigma(p \to p_i) + (r \to q) + \Sigma(r \to r_k) = 1$, and so if $r \in \mathrm{SP}\{p_i\}$ we obtain $(r \to q) = 0$, i.e., $r \in (\{p_i\}^\perp)^\perp$. ■

Lemma 3. Let $q_j \in \mathrm{SP}\{p_i\}$, and $\{q_j\}$ orthogonal. Then $\mathrm{SP}\{q_j\} \subseteq \mathrm{SP}\{p_i\}$.

Proof. Consider an element $r \in \{p_i\}^\perp$; by Lemma 2, r is orthogonal to every element in $\mathrm{SP}\{p_i\}$, hence r is orthogonal to all q_j. Therefore, again by Lemma 2, r is in $(\mathrm{SP}\{q_j\})^\perp$. So we have established that $q \in \mathrm{SP}\{q_j\}$ implies q orthogonal to $\{p_i\}^\perp$, i.e., $q \in (\{p_i\}^\perp)^\perp = \mathrm{SP}\{p_i\}$.

Lemma 4. Given $\{p_i\}$ and $\{q_j\}$ such that $\{p_i, q_j\}$ is maximal orthogonal in $\mathfrak{M}$, we have $\mathrm{SP}\{p_i\}^\perp = \mathrm{SP}\{q_j\}$.

Proof. Consider any $q \in \mathrm{SP}\{q_j\}$; since $\Sigma(q \to p_i) + \Sigma(q \to q_j) = 1$ and also $\Sigma(q \to q_j) = 1$, we have $(q \to p_i) = 0$, and thus $q \in \{p_i\}^\perp$, i.e., $q \in \mathrm{SP}\{p_i\}^\perp$. Conversely, let $r \in \mathrm{SP}\{p_i\}^\perp$; since $\Sigma(r \to p_i) + \Sigma(r \to q_j) = 1$, we obtain $\Sigma(r \to q_j) = 1$, i.e., $r \in \mathrm{SP}\{q_j\}$. ■

Lemma 5. If $\{r_k\}$ is maximal orthogonal in $\mathrm{SP}\{p_i\}$, then $\mathrm{SP}\{r_k\} = \mathrm{SP}\{p_i\}$.

Proof. Choose $\{q_j\}$ so that $\{p_i, q_j\}$ is maximal orthogonal in $\mathfrak{M}$. Since $\mathrm{SP}\{q_j\} = \mathrm{SP}\{p_i\}^\perp$, we have q_j orthogonal to each r_k. Thus $\{q_j, r_k\}$ is an orthogonal family. But if r is orthogonal to all q_j, r_k, then $r \in \mathrm{SP}\{q_j\}^\perp = \mathrm{SP}\{p_i\}$ (by Lemma 4) hence $\{r, r_k\}$ is an orthogonal family in $\mathrm{SP}\{p_i\}$ which is impossible since $\{r_k\}$ is maximal orthogonal in $\mathrm{SP}\{p_i\}$. Thus $\{r_k, q_j\}$ is maximal orthogonal in $\mathfrak{M}$, hence $\mathrm{SP}\{r_k\} = \mathrm{SP}\{q_j\}^\perp = \mathrm{SP}\{p_i\}$. ■

Lemma 6. Let $\{p_i\}$, $\{q_j\}$ be orthogonal families. Then $\mathrm{SP}\{p_i\} = \mathrm{SP}\{q_j\}$ iff for any $r \in \mathfrak{M}$ we have $\Sigma(r \to p_i) = \Sigma(r \to q_j)$.

Proof. The condition is obviously sufficient. So let $\mathrm{SP}\{p_i\} = \mathrm{SP}\{q_j\}$ and select a maximal orthogonal family $\{r_k\}$ in $\mathrm{SP}\{p_i\}^\perp (= \mathrm{SP}\{q_j\}^\perp)$. Then both $\{p_i, r_k\}$ and $\{q_j, r_k\}$ are maximal orthogonal in $\mathfrak{M}$, hence for all $r \in \mathfrak{M}$ we have $\Sigma(r \to p_i) + \Sigma(r \to r_k) = 1 = \Sigma(r \to q_j) + \Sigma(r \to r_k)$ which yields the desired condition. ■

This is as good a place as any to verify that if two rows are identical, i.e., if $(p \to r) = (q \to r)$ for all r, then the corresponding columns also are identical, i.e., $(r \to p) = (r \to q)$ for all r. To see this note that the hypothesis implies $(p \to q) = (q \to p) = 1$. This is enough to produce the desired result. Because $(p \to q) = 1$ implies $p \in \mathrm{SP}\{q\}$ (note that $\{q\}$ is an orthogonal set) hence $\mathrm{SP}\{p\} \subseteq \mathrm{SP}\{q\}$. But then $(q \to p) = 1$ similarly gives $\mathrm{SP}\{q\} \subseteq \mathrm{SP}\{p\}$. Thus $\mathrm{SP}\{p\} = \mathrm{SP}\{q\}$ and Lemma 6 completes the argument.

Remark. The definition of a span may seem somewhat awkward as far as verification or construction goes, particularly since it seems that knowledge of all values of the functional $(p \to q)$ is necessary. It turns out that all one needs is knowledge of all orthogonal pairs. This is essentially contained in the above lemmas, in particular Lemma 4: a subset of $\mathfrak{M}$ is a span iff it is the orthocomplement of some orthogonal set. It is not, however, possible to define arbitrarily [subject to condition (ii)] the various orthogonal pairs.

5. CONSTRUCTION OF THE LOGIC

Given a generalized Mielnik system, let $\mathfrak{L}$ be the set of all spans. For $A, B \in \mathfrak{L}$ let $A \leqslant B$ mean inclusion $\subseteq$, and let A' be the span $A^{\perp}$ (by Lemma 4, $A^{\perp}$ is indeed a span).

Theorem 3. The set $\mathfrak{L}$ with the above structure is a logic, i.e., axioms 1 through 4 hold.

Proof. It is clear that $\leqslant$ is a partial order and that $'$ satisfies Axiom 1. The element 0 is just $\varnothing$ and the element I is $\mathfrak{M}$, and Axiom 2 is also quite clearly valid. To verify the orthomodular law, let $A \leqslant B$, $A = \mathrm{SP}\{p_i\}$ and select $\{q_j\}$ so that $\{p_i, q_j\}$ are maximal orthogonal in B. We then have $B = \mathrm{SP}\{p_i, q_j\}$ by Lemma 5. Let $C = \mathrm{SP}\{q_j\}$ and note that $C \leqslant B$ and that since all q_j are orthogonal to the p_i, we have $C \leqslant A'$. To show that $C = A' \wedge B$, consider any $D \leqslant A'$, $D \leqslant B$. For any $r \in D$ we have r orthogonal to all p_i, since $p_i \in A$; on the other hand $r \in B$ implies $\Sigma(r \to p_i) + \Sigma(r \to q_j) = 1$, and so $\Sigma(r \to q_j) = 1$, i.e., $r \in C$. So $D \leqslant C$, which means that $C = A' \wedge B$. To show that $B = A \vee C$ we consider some $E \geqslant A$, $E \geqslant C$; since $p_i, q_j \in E$ we have $\mathrm{SP}\{p_i, q_j\} \subseteq E$, i.e., $B \leqslant E$, which means B is indeed $A \vee C$. Finally we verify Axiom 4. Let $A_i = \mathrm{SP}\{p_{ij} | j \in J_i\}$, and let A_{i_1}, A_{i_2} be disjoint, which means that the whole family $\{p_{ij}\}$ is orthogonal. Write A for $\mathrm{SP}\{p_{ij}\}$. Evidently $A_i \leqslant A$, and so we consider some $B \geqslant A_i$; since all $p_{ij} \in B$ we can find $r_k \in B$ so that $\mathrm{SP}\{p_{ij}, r_k\} = B$. But then, $r \in A$ implies $\Sigma(r \to p_{ij}) = 1$, hence also $\Sigma(r \to p_{ij}) + \Sigma(r \to r_k) = 1$, since this sum is $\leqslant 1$ by property (iii) and $\geqslant 1$ by the previous relation. Thus $r \in B$, i.e., $A \leqslant B$, and this means A is the supremum of the A_i. ∎

For any $p \in \mathfrak{M}$, $A \in \mathfrak{L}$ we define pA to be the number $\Sigma(p \to p_i)$, where $A = \mathrm{SP}\{p_i\}$. By Lemma 6 this does not depend on the choice of the maximal orthogonal set $\{p_i\}$ in A. It is clear from the structure of ΣA_j described in the last proof that the map $A \to pA$ is a state of $\mathfrak{L}$. Note that if $pA = qA$ for all A, then $p = q$; because this implies $(p \to r) = (q \to r)$ for all r, and so by the remarks in Section 4 we have $p = q$. We shall thus identify the map $A \to pA$ to the element p.

Theorem 4. The set $\mathfrak{M}$, considered as a set of states of $\mathfrak{L}$, satisfies Axioms 5 and 6.

Proof. Note that $pA = 1$ iff $p \in A$; this makes Axiom 5 obvious. Now given $p \in \mathfrak{M}$, consider the element $\mathrm{SP}\{p\} = \{q \in \mathfrak{M} | (q \to p) = 1\}$; evidently $p(\mathrm{SP}\{p\}) = 1$. Now if $pA = 1$, then $p \in A$ and (by Lemma 3) $\mathrm{SP}\{p\} \subseteq A$. Therefore $\mathrm{SP}\{p\}$ is the support of the state p. Finally, if $A \neq 0$ it has the form $\mathrm{SP}\{p_i\}$ for some $\{p_i\}$; but clearly $p_i A = 1$, and so Axiom 6 holds. ■

The question of whether $\mathfrak{L}$ is a lattice can be answered at once by the following.

Proposition 4. If A is the infimum of $\{A_i\}$, then $A = \cap A_i$ (as sets). Thus a family $\{A_i\}$ has an infimum iff its set intersection is a span.

Proof. Evidently $A \subseteq \cap A_i$. Now if $\{q_j\}$ is any orthogonal set in $\cap A_i$, we have $\mathrm{SP}\{q_j\} \leqslant A_i$ for all i, hence $\mathrm{SP}\{q_j\} \leqslant A$. But every element of A_i is part of an orthogonal set, hence $\cap A_i \subseteq A$. The rest is obvious. ■

It is not hard to see that it suffices to have $\cap A_i$ a subspace, in the sense that the functional $(p \to q)$ restricted to $\cap A_i$ satisfies the basic properties (i), (ii), (iii) in Section 4.

Not all logics constructed in this way will be lattices, even if the functional $(p \to q)$ is symmetric, as the following example shows:

$\to$	p	q	r	s	u	v	x	y
p	1	0	0	α	$1-\alpha$	α	0	0
q	0	1	0	$1-\alpha$	α	γ	$1-\gamma-\alpha$	$\alpha+\gamma$
r	0	0	1	0	0	$1-\gamma-\alpha$	$\gamma+\alpha$	$1-\gamma-\alpha$
s	α	$1-\alpha$	0	1	0	$\gamma+\alpha$	$1-\gamma-\alpha$	γ
u	$1-\alpha$	α	0	0	1	0	0	α
v	α	γ	$1-\gamma-\alpha$	$\gamma+\alpha$	0	1	0	$1-\alpha$
x	0	$1-\gamma-\alpha$	$\gamma+\alpha$	$1-\gamma-\alpha$	0	0	1	0
y	0	$\alpha+\gamma$	$1-\gamma-\alpha$	γ	α	$1-\alpha$	0	1

Indeed: $\mathrm{SP}\{p, q\} = \{p, q, s, u\}$, $\mathrm{SP}\{p, y\} = \{p, u, v, y\}$ and their intersection is $\{p, u\}$, which is not a span.

6. REMARKS

It is of importance to known whether the relaxation of symmetry on the functional $(p \to q)$ has any significance, i.e., whether it actually produces more logics. In the Appendix we show that a certain logic generated by six states cannot be obtained from a symmetric functional. Thus we have indeed a more extensive class.

It is not difficult to see that asymmetric functionals of any "size" (finite or infinite) can be constructed. As yet, we do not have a systematic way of constructing them all.

APPENDIX

Consider the logic $\mathfrak{L}$ determined by the system

$\rightarrow$	p	q	r	s	u	v
p	1	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
q	0	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$
r	0	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$
s	0	0	0	1	0	0
u	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	1	0
v	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	1

There are two maximal orthogonal sets $\{p,q,r,s\}$ and $\{s,u,v\}$. Besides the singleton spans, there are eight two-element spans: $\{p,q\}$, $\{p,r\}$, $\{p,s\}$, $\{q,r\}$, $\{q,s\}$, $\{r,s\}$, $\{s,u\}$, $\{s,v\}$; three three-element spans: $\{p,q,s\}$, $\{p,r,s\}$, $\{q,r,s\}$; and one five-element span: $\{p,q,r,u,v\}$. Counting 0 and I we have 20 elements in all. The six one-element spans are the atoms of $\mathfrak{L}$, and we shall write X for the span $\mathrm{SP}\{x\}$. Thus a state m of $\mathfrak{L}$ is determined by the 6-tuple $(m(P), m(Q), m(R), m(S), m(U), m(V))$ subject to the conditions $m(X)\geqslant 0$, $m(P)+m(Q)+m(R)+m(S)=1$, $m(S)+m(U)+m(V)=1$. It is straightforward to verify that there are seven pure states $(0,0,0,1,0,0)$, $(1,0,0,0,0,1)$, $(0,1,0,0,0,1)$, $(0,0,1,0,0,1)$, $(1,0,0,0,1,0)$, $(0,1,0,0,1,0)$, $(0,0,1,0,1,0)$ so that the arbitrary state of $\mathfrak{L}$ has the form $(\lambda_2+\lambda_5, \lambda_3+\lambda_6, \lambda_4+\lambda_7, \lambda_1, \lambda_5+\lambda_6+\lambda_7, \lambda_2+\lambda_3+\lambda_4)$, where $\lambda_i\geqslant 0$, $\Sigma_{i=1}^{7}\lambda_i=1$. We shall show that there is no "rich" subset for which the functional $(p\rightarrow q)=p(L_q)$ is symmetric. Assume there is. Then the second part of Axiom 6 implies that the "rich" set must include one state m for which $m(P)=1$, i.e., of the form $(1,0,0,0,\alpha,1-\alpha)$; similarly, taking all other atoms into account, we see that states of the forms $(0,1,0,0,\beta,1-\beta)$, $(0,0,1,0,\gamma,1-\gamma)$, $(0,0,0,1,0,0)$, $(\lambda,\mu,1-\lambda-\mu,0,1,0)$, $(\rho,\tau,1-\rho-\tau,0,0,1)$ must be included corresponding to Q,R,S,U,V, respectively, where $\alpha,\beta,\gamma,\lambda,\mu,\rho,\tau$ are different from 0 and 1. Note that each of these states has the corresponding atom as a support. The transition probability matrix for these states thus obtains the following form:

$$\begin{matrix} 1 & 0 & 0 & 0 & \alpha & 1-\alpha \\ 0 & 1 & 0 & 0 & \beta & 1-\beta \\ 0 & 0 & 1 & 0 & \gamma & 1-\gamma \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \lambda & \mu & 1-\lambda-\mu & 0 & 1 & 0 \\ \rho & \tau & 1-\rho-\tau & 0 & 0 & 1 \end{matrix}$$

and it is clear that no choice of the parameters can make it symmetric. Since any "rich" system of states must include the above, we see that no rich symmetric systems of states exists.